The M-Algebra

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ABSTRACT

We construct a new extension of the Poincaré superalgebra in eleven dimensions which contains super one-, two- and five-form charges. The latter two are associated with the supermembrane and the superfivebrane of M-theory. Using the Maurer-Cartan equations of this algebra, we construct closed super seven-forms in a number of ways. The pull-back of the corresponding super six-forms are candidate superfivebrane Wess-Zumino terms, which are manifestly supersymmetric, and contain coordinates associated with the new charges.

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As is well known, supergravity theories in diverse dimensions admit a variety of super p-brane solitons. It is now widely appreciated that these objects play an important role in nonperturbative string physics. While it is difficult to analyze the dynamics of super p-branes in general, they do possess some algebraic properties that are more amenable to study. One of these properties is the modification of the Poincaré superalgebra in presence of super p-brane solitons [1, 2].

Let us consider the case of eleven dimensional supergravity. It admits supermembrane soliton [3] which modifies the 11D Poincaré superalgebra as follows:

$$\{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta} \, P_\mu + \gamma_{\mu\nu\alpha\beta} \, Z^{\mu\nu} \, , \tag{1}$$

where the topological charge $Z^{\mu\nu}$ arises due to the presence of a supermembrane configuration in spacetime. The Poincaré supercharges $Q_A = (P_\mu, Q_\alpha)$ generate supertranslations in flat superspace parametrized by the coordinates $Z^M = (X^\mu, \theta^\alpha)$, $\mu = 0, 1, ..., 10$, where θ^α are anticommuting 32 component Majorana spinors. The notation for the Dirac γ-matrices is self explanatory.

Given a supermembrane soliton in $D = 11$, the Noether supercharge per unit membrane area is defined as an integral over an eight dimensional transverse space-like surface. The Poisson bracket algebra of the supercharges yields the result (1). The p-form charges arising in this way are variably referred to as topological charges, or Page charges, or sometimes as central charges. Strictly speaking, they are not central because they do not commute with Lorentz generators, except for $p = 0$.

The occurrence of the topological charge $Z^{\mu\nu}$ in 11D Poincaré superalgebra can also be understood from the supermembrane worldvolume point of view. As shown in [2], the presence of a Wess-Zumino term in the supermembrane action, which is supersymmetric only up to a total derivative term, modifies the algebra of supercharges precisely as in (1), with the topological charge realized as

$$Z^{\mu\nu} = \int d^2\sigma \, \epsilon^{0ij} \partial_i X^\mu \partial_j X^\nu \, , \tag{2}$$

where $\sigma^i \, (i = 0, 1, 2)$ are the worldvolume coordinates. These can also be viewed as the topological charges associated with the identically conserved topological current $J_T^{i\mu\nu} = \epsilon^{ijk} \partial_j X^\mu \partial_k X^\nu$. If at fixed time the membrane defines a nontrivial 2-cycle in space, then the above integral will be nonzero [2].

It is known that $D = 11$ supergravity also admits a superfivebrane soliton [4]. Although the corresponding superfivebrane action has yet to be constructed, it is reasonable to expect that five-form topological charges will be associated with them.

Recently, it has been argued that [5] the p-brane topological charges discussed above are closely associated with the boundaries of p-branes and as well as the topology of the background geometry

in which they propagate. It is furthermore observed [5] that more general kinds of topological charges may emerge in general backgrounds other than flat spacetime.

Sometime ago, Bergshoeff and the author [6, 7] indeed found an extension of the 11D Poincaré superalgebra with the extra generators $Z^{\mu\nu}$ and $Z^{\mu\alpha}$. A Kac-Moody extension of the algebra naturally led to the following generators:

$$Z^{\mu\nu}(\sigma) = \epsilon^{0ij}\, L_j^\mu L_k^\nu\ , \qquad Z^{\mu\beta}(\sigma) = \epsilon^{0ij}\, L_j^\mu L_k^\beta\ , \qquad Z^{\alpha\beta}(\sigma) = \epsilon^{0ij}\, L_j^\alpha L_k^\beta\ , \tag{3}$$

where σ refers to the membrane worldvolume coordinates, and the supersymmetric line elements are given by

$$L_i^\mu = \partial_i X^\mu + \tfrac{1}{2}\gamma^\mu_{\alpha\beta}\,\theta^\alpha\partial_i\theta^\beta\ , \qquad L_i^\alpha = \partial_i\theta^\alpha\ . \tag{4}$$

Later, the extended version of 11D Poincaré superalgebra found in [6] was generalized to include the generators $Z^{\alpha\beta}$ [8], and a spinor generator Z^α [9]. As far as we know, there is no superstring soliton in $D = 11$, and therefore the occurrence of the spinor generator Z^α is somewhat of a mystery at present (see, however, [10, 9]). Its occurrence in 10D Poincaré superalgebra is natural, and indeed Green [11] discovered it sometime ago. This superalgebra was used in an interesting way by Siegel [12], who introduced a new coordinate for the extra fermionic generator, and reformulated the Green-Schwarz superstring such that the full action, including the Wess-Zumino term, exhibited manifest supersymmetry. Siegel's result forms an important part of our motivation for this work, and therefore we shall come back to this point later.

Interestingly enough, extensions of the 11D Poincaré superalgebra which contains some of the generators mentioned above were considered long ago, before the discovery of higher super p-branes. In particular, D'Auria and Fré [13] considered the extra generators $Z^{\mu\nu}$, $Z^{\mu_1\cdots\mu_5}$ and Z^α, in their attempts to gauge the $D = 11$ supergravity. The issue of whether the dual formulation of $D = 11$ supergravity with a six-form potential existed also emerged in a related study which used these extensions [14].

It is clearly of interest to unify and extend the results mentioned above, in a way that would take into account the existence of the superfivebrane in $D = 11$. Further motivations for studying the general topological extensions of 11D Poincaré superalgebra are: (a) They may provide a powerful tool, in the framework of an extended super Poincaré geometry with new p-form coordinates, for probing supermembrane-superfivebrane duality, (b) the topologically extended super Poincaré geometry may provide important ingredients for the construction of the elusive eleven dimensional superfivebrane action and (c) knowledge of their representations may shed some light some algebraic aspects of M-theory, including the spectrum of nonperturbative states [5].

In this paper, we shall give the most general extension of the 11D Poincaré superalgebra motivated by some geometrical considerations which will be spelled out below, and that contains super one-, two- and five-form charges

$$Z^{\mu_1\cdots\mu_k\alpha_{k+1}\cdots\alpha_p} , \quad p = 1, 2, 5 ; \; k = 0, 1, ..., p , \tag{5}$$

in addition to the usual super Poincaré generators P_μ and Q_α. Our result contains the ones in [6, 8, 9, 13] as special cases. For short, we will refer to this algebra as the M-algebra. We have in mind, of course, the role it is expected to play in M-theory. Interestingly, we find that the existence of the super five-form charges in the algebra necessarily requires the presence of the super two-form charges, while the reverse is not true. Moreover, it turns out that some of the super two-form charges cease to (anti) commute with each other.

We expect that there will be a number of interesting properties of the M-algebra which will be uncovered in the future. For the purposes of this note, however, it will suffice to (a) present the algebra, and (b) to construct closed super seven-forms that live on the full supergroup manifold. At the end of the paper, we shall comment on their possible use in the construction of candidate Wess-Zumino terms for the eleven dimensional superfivebrane.

We now turn to the description of the M-algebra. Let us denote the generators of the algebra collectively as $T_{\hat{A}}$. We consider the generators

$$T_{\hat{A}} = (Q_A, \; Z^A, \; Z^{AB}, \; Z^{A_1\cdots A_5}) , \qquad Q_A = (P_\mu, Q_\alpha) , \tag{6}$$

where the Z-generators are totally graded antisymmetric. The M-algebra can be written as

$$[T_{\hat{A}}, T_{\hat{B}}\} = f_{\hat{A}\hat{B}}{}^{\hat{C}} \, T_{\hat{C}} , \tag{7}$$

where the structure constants will be given shortly. In the dual basis, one defines the Maurer-Cartan super one-forms

$$e^{\hat{A}} = dZ^{\hat{M}} \, L_{\hat{M}}{}^{\hat{A}} , \tag{8}$$

where $dZ^{\hat{M}}$ are the differentials on the supergroup manifold based on the M-algebra. We can also define the supersymmetric line elements

$$L_i^A = \partial_i Z^{\hat{M}} L_{\hat{M}}{}^{\hat{A}} . \tag{9}$$

Explicit expressions for these objects can be obtained straightforwardly from

$$U^{-1}\partial_i U = L_i^{\hat{A}} \, T_{\hat{A}} , \tag{10}$$

where U is a group element, which can be parametrized in terms of the usual superspace coordinates, and the new ϕ-coordinates associated with the Z-generators as follows:

$$U = e^{\phi_{\mu_1\cdots\mu_5}Z^{\mu_1\cdots\mu_5}}\cdots e^{\phi_{\alpha_1\cdots\alpha_5}Z^{\alpha_1\cdots\alpha_5}}\ e^{\phi_{\mu\nu}Z^{\mu\nu}}\cdots e^{\phi_{\alpha\beta}Z^{\alpha\beta}}\ e^{\phi_\mu Z^\mu}\ e^{\phi_\alpha Z^\alpha}\ e^{X^\mu P_\mu}\ e^{\theta^\alpha Q_\alpha}\ . \tag{11}$$

The details of $L_i^{\hat{A}}$, as well as the M-group transformations under which they are invariant, are not particularly illuminating. However, the interested reader can find useful formulae for their computation, as well as explicit expressions for $L_i^{\hat{A}}$ in the case of Q_A and Z^{AB} generators in [8].

As is well known, the Maurer-Cartan structure equations [2]

$$de^{\hat{A}} = -\tfrac{1}{2}\ e^{\hat{B}} \wedge e^{\hat{C}}\ f_{\hat{C}\hat{B}}{}^{\hat{A}}\ , \tag{12}$$

contain equivalent information about the algebra. The fact that the Jacobi identities are satisfied is, of course, encoded in the integrability condition $d^2 e^{\hat{A}} = 0$. It is convenient to present our results first in the form of Maurer-Cartan equations. The strategy we have followed to determine these equations is very simple: We have parametrized the algebra in the most general possible way that contains the components of the following super forms as structure constants:

$$T^\mu \;\; = \;\; -\tfrac{1}{2}e^\alpha \wedge e^\beta\ \gamma^\mu_{\alpha\beta}\ , \tag{13}$$

$$H_3^{(0)} \;\; = \;\; e^\mu \wedge e^\alpha \wedge e^\beta\ \gamma_{\mu\alpha\beta}\ , \tag{14}$$

$$H_4 \;\; = \;\; \tfrac{1}{4}e^\mu \wedge e^\nu \wedge e^\alpha \wedge e^\beta\ \gamma_{\mu\nu\alpha\beta}\ , \tag{15}$$

$$H_7^{(0)} \;\; = \;\; \tfrac{1}{5!}e^{\mu_1} \wedge \cdots \wedge e^{\mu_5} \wedge e^\alpha \wedge e^\beta\ \gamma_{\mu_1\cdots\mu_5\alpha\beta}\ . \tag{16}$$

Noting that the structure equations for e^μ and e^α will not be modified compared to those in ordinary Poincaré superspace, one can verify that the four-form H_4 is closed, thanks to the following well known identity

$$\gamma_{\mu\nu(\alpha\beta}\,\gamma^\nu_{\gamma\delta)} = 0\ , \tag{17}$$

which holds in $D = 4, 5, 7, 11$ [17]. The super-forms $H_3^{(0)}$ and $H_7^{(0)}$ are not closed. However, as we will see later, they can be modified so as to be closed in the full M-algebra.

The occurrence of (13) and (15) can be understood from the structure of the known super p-brane actions [6, 7, 16]. The inclusion of (16) is motivated by superfivebrane considerations, and the assumption that there may exist a dual formulation of 11D supergravity in which, both, the three-form and six-form potentials occur. Finally, we have included (14) for the sake of completeness; a point which will become more transparent below.

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[2]Our conventions for super p-forms are those of [15]. In particular, the components of a super p-form F are defined by $F = \tfrac{1}{p!}\ e^{M_1} \wedge \cdots e^{M_p}\ F_{M_p\cdots M_1}$, and the exterior derivative by $dF = \tfrac{1}{p!}\ e^{M_1} \wedge \cdots e^{M_p} \wedge e^M\ \partial_M F_{M_p\cdots M_1}$. Furthermore, given a super q-form G, we have the rule: $d(F \wedge G) = F \wedge dG + (-1)^q\ dF \wedge G$.

The parametrization of the ansatze in a manner described above contains a large number of parameters. We have determined these parameters by explicit computation of all the integrability conditions $d^2 e^{\hat{A}} = 0$. These conditions not only are sufficient to determine all the parameters in our ansatze, but they also provide several cross checks, since we obtain an overdetermined system of equations for these parameters. It is also worth mentioning that, in addition to repeated use of (17), we have also used the following identity [14]

$$\gamma_{\lambda(\alpha\beta}\gamma^{\lambda\mu\nu\rho\sigma}_{\gamma\delta)} - 3\gamma^{[\mu\nu}_{(\alpha\beta}\gamma^{\rho\sigma]}_{\gamma\delta)} = 0 \; . \tag{18}$$

Also useful are the following identities

$$\gamma_{\mu(\alpha\beta}\gamma^{\mu}_{\gamma\delta)} + \tfrac{1}{10}\gamma_{\mu\nu(\alpha\beta}\gamma^{\mu\nu}_{\gamma\delta)} = 0 \; , \tag{19}$$

$$\gamma_{\mu(\alpha\beta}\gamma^{\mu}_{\gamma\delta)} + \tfrac{1}{720}\gamma_{\mu_1\cdots\mu_5(\alpha\beta}\gamma^{\mu_1\cdots\mu_5}_{\gamma\delta)} = 0 \; . \tag{20}$$

The identities (18) and (19) follow straightforwardly from the main identity (17), and the identity (20) can be easily derived from (18).

With the preliminaries thus explained, we are now ready to present our results. We propose the following set of Maurer-Cartan equations:

$$
\begin{aligned}
de^{\mu} &= -\tfrac{1}{2}\, e^{\alpha} \wedge e^{\beta}\, \gamma^{\mu}_{\alpha\beta} \; , \\
de^{\alpha} &= 0 \; , \\
de'_{\mu} &= -\tfrac{1}{2}e^{\alpha} \wedge e^{\beta}\, \gamma_{\mu\alpha\beta} \; , \\
de'_{\alpha} &= -e^{\beta} \wedge e^{\mu}\, \gamma_{\mu\alpha\beta} + (1-\lambda-\tau)\, e^{\beta} \wedge e'_{\mu}\, \gamma^{\mu}_{\alpha\beta} - \tfrac{\lambda}{10}\, e^{\beta} \wedge e_{\mu\nu}\, \gamma^{\mu\nu}_{\alpha\beta} \\
&\quad -\tfrac{\tau}{720}\, e^{\beta} \wedge e_{\mu_1\cdots\mu_5}\, \gamma^{\mu_1\cdots\mu_5}_{\alpha\beta} \; , \\
de_{\mu\nu} &= -\tfrac{1}{2}e^{\alpha} \wedge e^{\beta}\, \gamma_{\mu\nu\alpha\beta} \; , \\
de_{\mu\alpha} &= -e^{\beta} \wedge e^{\nu}\, \gamma_{\mu\nu\alpha\beta} - e^{\beta} \wedge e_{\mu\nu}\, \gamma^{\nu}_{\alpha\beta} \; , \\
de_{\alpha\beta} &= \tfrac{1}{2}e^{\mu} \wedge e^{\nu}\, \gamma_{\mu\nu\alpha\beta} - \tfrac{1}{2}e_{\mu\nu} \wedge e^{\mu}\, \gamma^{\nu}_{\alpha\beta} - \tfrac{1}{4}e_{\mu\gamma} \wedge e^{\gamma}\, \gamma^{\mu}_{\alpha\beta} - 2e_{\mu\alpha} \wedge e^{\gamma}\, \gamma^{\mu}_{\beta\gamma} \; , \\
de_{\mu_1\cdots\mu_5} &= -\tfrac{1}{2}e^{\alpha} \wedge e^{\beta}\, \gamma_{\mu_1\cdots\mu_5\alpha\beta} \; , \\
de_{\mu_1\cdots\mu_4\alpha} &= e^{\beta} \wedge e^{\tau}\, \gamma_{\tau\mu_1\cdots\mu_4\alpha\beta} + e^{\beta} \wedge e_{\tau\mu_1\cdots\mu_4}\, \gamma^{\tau}_{\alpha\beta} - 6e^{\beta} \wedge e_{\mu_1\mu_2}\, \gamma_{\mu_3\mu_4\alpha\beta} \; , \\
de_{\mu\nu\rho\alpha\beta} &= \tfrac{1}{2}e^{\sigma} \wedge e^{\tau}\, \gamma_{\sigma\tau\mu\nu\rho\alpha\beta} - \tfrac{1}{2}e_{\mu\nu\rho\sigma\tau} \wedge e^{\sigma}\, \gamma^{\tau}_{\alpha\beta} + \tfrac{1}{4}e_{\mu\nu\rho\tau\gamma} \wedge e^{\gamma}\, \gamma^{\tau}_{\alpha\beta} \\
&\quad -2e_{\tau\mu\nu\rho\alpha} \wedge e^{\gamma}\, \gamma^{\tau}_{\beta\gamma} + \tfrac{3}{2}e^{\tau} \wedge e_{\tau\mu}\, \gamma_{\nu\rho\alpha\beta} - 3e^{\tau} \wedge e_{\mu\nu}\, \gamma_{\rho\tau\alpha\beta} \\
&\quad -\tfrac{3}{4}e^{\gamma} \wedge e_{\mu\gamma}\, \gamma_{\nu\rho\alpha\beta} - 6e^{\gamma} \wedge e_{\mu\alpha}\, \gamma_{\nu\rho\beta\gamma} + \tfrac{3}{2}e_{\tau\mu} \wedge e_{\nu\rho}\, \gamma^{\tau}_{\alpha\beta} \; , \\
de_{\mu\nu\alpha\beta\gamma} &= 2e^{\sigma} \wedge e_{\sigma\tau\mu\nu\alpha}\, \gamma^{\tau}_{\beta\gamma} + e^{\delta} \wedge e_{\tau\mu\nu\delta\alpha}\, \gamma^{\tau}_{\beta\gamma} + 5e^{\delta} \wedge e_{\tau\mu\nu\alpha\beta}\, \gamma^{\tau}_{\gamma\delta} \\
&\quad -10e^{\tau} \wedge e_{\mu\alpha}\, \gamma_{\nu\tau\beta\gamma} - 5e^{\delta} \wedge e_{\alpha\beta}\, \gamma_{\mu\nu\gamma\delta} - 2e^{\tau} \wedge e_{\tau\alpha}\, \gamma_{\mu\nu\beta\gamma}
\end{aligned}
$$

$$-e^\delta \wedge e_{\delta\alpha} \; \gamma_{\mu\nu\beta\gamma} - 6e_{\tau\mu} \wedge e_{\nu\alpha} \; \gamma^\tau_{\beta\gamma} - 2e_{\tau\alpha} \wedge e_{\mu\nu} \; \gamma^\tau_{\beta\gamma} \; ,$$

$$de_{\mu\alpha_1\cdots\alpha_4} = -e^\nu \wedge e_{\mu\nu\tau\alpha_1\alpha_2} \; \gamma^\tau_{\alpha_3\alpha_4} - \tfrac{3}{10}e^\gamma \wedge e_{\mu\nu\gamma\alpha_1\alpha_2} \; \gamma^\nu_{\alpha_3\alpha_4} - \tfrac{6}{5}e^\gamma \wedge e_{\mu\nu\alpha_1\alpha_2\alpha_3} \; \gamma^\nu_{\alpha_4\gamma}$$

$$+3e^\nu \wedge e_{\alpha_1\alpha_2} \; \gamma_{\mu\nu\alpha_3\alpha_4} + 2e_{\mu\nu} \wedge e_{\alpha_1\alpha_2} \; \gamma^\nu_{\alpha_3\alpha_4} + 3e_{\nu\alpha_1} \wedge e_{\mu\alpha_2} \; \gamma^\nu_{\alpha_3\alpha_4} \; ,$$

$$de_{\alpha_1\cdots\alpha_5} = -\tfrac{3}{5}e^\mu \wedge e_{\mu\nu\alpha_1\alpha_2\alpha_3} \; \gamma^\nu_{\alpha_4\alpha_5} + e^\gamma \wedge e_{\mu\gamma\alpha_1\alpha_2\alpha_3} \; \gamma^\mu_{\alpha_4\alpha_5} + \tfrac{7}{2}e^\gamma \wedge e_{\mu\alpha_1\cdots\alpha_4} \; \gamma^\mu_{\alpha_5\gamma}$$

$$+6e_{\mu\alpha_1} \wedge e_{\alpha_2\alpha_3} \; \gamma^\mu_{\alpha_4\alpha_5} \; , \tag{21}$$

where it is understood that the obvious symmetries of indices on the left hand side are to be implemented on the right hand side, with unit strength (anti) symmetrizations. The parameters λ and τ are arbitrary. Of course, by rescaling various one-forms, one can introduce a number of new parameters. However, the consistency of any contraction has to be checked carefully. We shall come back to this point shortly.

The world indices can be raised and lowered with the 11D Minkowski metric, and the fermionic indices with the charge–conjugation matrix, as usual. However, we have found it convenient not to do so in our calculations, and to always keep the world and spinor indices in a fixed position (see comment (iii) below). Nevertheless, to avoid any confusion between the one-forms associated with the supertranslation generators Q_A, and the topological charge Z^A, we have used a prime to distinguish the latter from the former.

To see the structure of the algebra that underlies the Maurer-Cartan equations (21) more explicitly, it is convenient to go over to the dual basis. This is easily done by using (7) and (12), and we thus find the following (anti) commutation rules:

$$\{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta} \; P_\mu + \gamma_{\mu\alpha\beta} \; Z^\mu + \gamma_{\mu\nu\alpha\beta} \; Z^{\mu\nu} + \gamma_{\mu_1\cdots\mu_5\alpha\beta} \; Z^{\mu_1\cdots\mu_5} \; ,$$

$$[P_\mu, Q_\alpha] = \gamma_{\mu\alpha\beta} \; Z^\beta - \gamma_{\mu\nu\alpha\beta} \; Z^{\nu\beta} - \gamma_{\mu\nu_1\cdots\nu_4\alpha\beta} \; Z^{\nu_1\cdots\nu_4\beta} \; ,$$

$$[P_\mu, P_\nu] = \gamma_{\mu\nu\alpha\beta} \; Z^{\alpha\beta} + \gamma_{\mu\nu\mu_1\cdots\mu_3\alpha\beta} \; Z^{\mu_1\cdots\mu_3\alpha\beta} \; ,$$

$$[Q_\alpha, Z^\mu] = (1 - \lambda - \tau) \; \gamma^\mu_{\alpha\beta} \; Z^\beta \; ,$$

$$[P_\lambda, Z^{\mu\nu}] = \tfrac{1}{2}\delta^\mu_\lambda \left(\gamma^\nu_{\alpha\beta} \; Z^{\alpha\beta} - 3\gamma_{\rho\sigma\alpha\beta} \; Z^{\nu\rho\sigma\alpha\beta} \right) + 3\gamma_{\lambda\rho\alpha\beta} \; Z^{\mu\nu\rho\alpha\beta} \; ,$$

$$[Q_\alpha, Z^{\mu\nu}] = -\tfrac{\lambda}{10}\gamma^{\mu\nu}_{\alpha\beta} \; Z^\beta + \gamma^\mu_{\alpha\beta} \; Z^{\nu\beta} - 6\gamma_{\rho\sigma\alpha\beta} \; Z^{\mu\nu\rho\sigma\beta} \; ,$$

$$[P_\mu, Z^{\nu\alpha}] = -2\delta^\nu_\mu \; \gamma_{\lambda\tau\beta\gamma} \; Z^{\lambda\tau\alpha\beta\gamma} + 10\gamma_{\mu\tau\beta\gamma} \; Z^{\nu\tau\alpha\beta\gamma} \; ,$$

$$\{Q_\alpha, Z^{\mu\beta}\} = \tfrac{1}{4}\delta^\beta_\alpha \; \gamma^\mu_{\gamma\delta} \; Z^{\gamma\delta} + 2\gamma^\mu_{\alpha\gamma} \; Z^{\beta\gamma} + \tfrac{3}{4}\delta^\beta_\alpha \; \gamma_{\nu\rho\gamma\delta} \; Z^{\mu\nu\rho\gamma\delta} + 6\gamma_{\nu\rho\alpha\gamma} \; Z^{\mu\nu\rho\beta\gamma} \; ,$$

$$[Q_\gamma, Z^{\alpha\beta}] = -5\gamma_{\mu\nu\gamma\delta} \; Z^{\mu\nu\alpha\beta\delta} - \delta^\alpha_\gamma \; \gamma_{\mu\nu\delta\epsilon} \; Z^{\mu\nu\beta\delta\epsilon} \; ,$$

$$[P_\mu, Z^{\alpha\beta}] = -3\gamma_{\mu\nu\gamma\delta} \; Z^{\nu\alpha\beta\gamma\delta} \; ,$$

$$[Z^{\mu\nu}, Z^{\rho\sigma}] = -3\gamma^\mu_{\alpha\beta} \; Z^{\nu\rho\sigma\alpha\beta} \; ,$$

$$[Z^{\mu\nu}, Z^{\rho\alpha}] = -6\gamma^\mu_{\beta\gamma} \; Z^{\nu\rho\alpha\beta\gamma} + 2\gamma^\rho_{\alpha\beta} \; Z^{\mu\nu\alpha\beta\gamma} \; ,$$

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$$\begin{aligned}
[Z^{\mu\nu}, Z^{\alpha\beta}] &= -2\gamma^{\mu}_{\gamma\delta}\, Z^{\nu\alpha\beta\gamma\delta} \ , \\
\{Z^{\mu\alpha}, Z^{\nu\beta}\} &= -3\gamma^{\mu}_{\gamma\delta}\, Z^{\nu\alpha\beta\gamma\delta} \ , \\
[Z^{\mu\alpha}, Z^{\beta\gamma}] &= 6\gamma^{\mu}_{\delta\epsilon}\, Z^{\alpha\beta\gamma\delta\epsilon} \ , \\
[Q_{\alpha}, Z^{\mu_1\cdots\mu_5}] &= -\tfrac{\tau}{720}\gamma^{\mu_1\cdots\mu_5}_{\alpha\beta}\, Z^{\beta} + \gamma^{\mu_5}_{\alpha\beta}\, Z^{\mu_1\cdots\mu_4\beta} \ , \\
[P_{\lambda}, Z^{\mu_1\cdots\mu_5}] &= \tfrac{1}{2}\delta^{\mu_1}_{\lambda}\, \gamma^{\mu_2}_{\alpha\beta}\, Z^{\mu_3\cdots\mu_5\alpha\beta} \ , \\
\{Q_{\alpha}, Z^{\beta\mu_1\cdots\mu_4}\} &= \tfrac{1}{4}\delta^{\beta}_{\alpha}\, \gamma^{\mu_1}_{\gamma\delta}\, Z^{\mu_2\cdots\mu_4\gamma\delta} + 2\gamma^{\mu_1}_{\alpha\delta}\, Z^{\mu_2\cdots\mu_4\beta\gamma} \ , \\
[P_{\lambda}, Z^{\beta\mu_1\cdots\mu_4}] &= 2\delta^{\mu_1}_{\lambda}\, \gamma^{\mu_2}_{\gamma\delta}\, Z^{\mu_2\mu_3\beta\gamma\delta} \ , \\
[Q_{\alpha}, Z^{\mu\nu\rho\beta\gamma}] &= \delta^{\beta}_{\alpha}\, \gamma^{\mu}_{\delta\epsilon}\, Z^{\nu\rho\gamma\delta\epsilon} + 5\gamma^{\mu}_{\alpha\epsilon}\, Z^{\nu\rho\beta\gamma\epsilon} \ , \\
[P_{\lambda}, Z^{\mu\nu\rho\beta\gamma}] &= -\delta^{\mu}_{\lambda}\, \gamma^{\nu}_{\delta\epsilon}\, Z^{\rho\beta\gamma\delta\epsilon} \ , \\
\{Q_{\delta}, Z^{\mu\nu\alpha\beta\gamma}\} &= -\tfrac{3}{10}\delta^{\alpha}_{\delta}\, \gamma^{\mu}_{\epsilon\kappa}\, Z^{\nu\beta\gamma\epsilon\kappa} - \tfrac{6}{5}\gamma^{\mu}_{\delta\epsilon}\, Z^{\nu\alpha\beta\gamma\epsilon} \ , \\
[P_{\lambda}, Z^{\mu\nu\alpha\beta\gamma}] &= -\tfrac{3}{5}\delta^{\mu}_{\lambda}\, \gamma^{\nu}_{\delta\epsilon}\, Z^{\alpha\beta\gamma\delta\epsilon} \ , \\
[Q_{\beta}, Z^{\mu\alpha_1\cdots\alpha_4}] &= \delta^{\alpha_1}_{\beta}\, \gamma^{\mu}_{\gamma\delta}\, Z^{\alpha_2\cdots\alpha_4\gamma\delta} + \tfrac{7}{2}\gamma^{\mu}_{\beta\gamma}\, Z^{\alpha_1\cdots\alpha_4\gamma} \ .
\end{aligned} \tag{22}$$

Several comments are in order:

(i) The existence of the algebra (22) is highly nontrivial. To show that the Jacobi identities are satisfied, one makes crucial use of the γ-matrix identity (17) and its consequences (18)-(20). As is well known, the identity (17) holds in $D = 4, 5, 7, 11$ [17], i.e. precisely the dimensions in which the supermembrane action of [15] exists.

(ii) While in the absence of the super five-form generators all the remaining Z-charges (anti) commute with each other, this ceases to be the case once the super five-form generators are introduced. This is a surprising feature, since one normally thinks of the topological charges as coming from antisymmetric products of L^{μ}_i and L^{α}_i, which are expected to have vanishing Poisson brackets with each other.

(iii) The form of the algebra is suggestive of a geometrization in which one works with the generators Q_A, Z^A, Z^{AB}, $Z^{A_1\cdots A_5}$ and use the super torsion tensor $T_{AB}{}^C$, and the graded antisymmetric tensors H_{ABC}, H_{ABCD} and $H_{A_1\cdots A_7}$ defined in (13)-(16) as structure constants. This would correspond to a rigid version of a curved superspace algebra. Surprisingly, this does not work, as it can already be established at the level of the subalgebra containing only the Z^{AB} as the new generators. This may suggest the existence of an improved version of the algebra which can be geometrized. Whether this is indeed possible remains to be seen.

(iv) The first line of this algebra can be put into a $10 + 2$ dimensional form

$$\{Q_{\alpha}, Q_{\beta}\} = \gamma_{\hat{\mu}\hat{\nu}\alpha\beta}\, Z^{\hat{\mu}\hat{\nu}} + \gamma_{\hat{\mu}_1\cdots\hat{\mu}_6\alpha\beta}\, Z^{\hat{\mu}_1\cdots\hat{\mu}_6}_{+} \ , \tag{23}$$

where $\hat{\mu} = 0, 1, ..., 10, 12$, Q_{α} is a 32 component Majorana-Weyl spinor, and the 66 component $Z^{\hat{\mu}\hat{\nu}}$,

together with the self-dual 462 component $Z_+^{\hat{\mu}_1 \cdots \hat{\mu}_6}$ add up to 528 generators. However, it is far from obvious if the full algebra presented above can be casted into a $10 + 2$ dimensional form.

(v) While the presence of the super two- and five-form generators are related to the existence of the supermembranes and superfivebranes of eleven dimensional supergravity, the occurrence of the super one-form generator Z^A is somewhat unexpected, and it is a surprising feature of the above algebra. See, however, [10, 9] where the issue of superstring in $D = 11$ is discussed. In particular, let us note the existence of the following closed super three-form

$$H_3 = e^\mu \wedge e^\alpha \wedge e^\beta \; \gamma_{\mu\alpha\beta} + e^\alpha \wedge e^\beta \left((\lambda + \tau - 1) \, e'_\mu \, \gamma^\mu_{\alpha\beta} + \tfrac{\lambda}{10} \; e_{\mu\nu}\gamma^{\mu\nu}_{\alpha\beta} + \tfrac{\tau}{720} \, e_{\mu_1 \cdots \mu_5} \, \gamma^{\mu_1 \cdots \mu_5}_{\alpha\beta} \right) \; . \quad (24)$$

Indeed $dH_3 = 0$, and expressing $H_3 = dC_2$, and using the Maurer-Cartan equations (21), one finds that

$$C_2 = -e^\alpha \wedge e'_\alpha \; . \quad (25)$$

One might envisage using the pull-back of this super two-form in constructing a Wess-Zumino term for a superstring action in $D = 11$. Interestingly enough, this form turns out to play a role in the construction of a novel Wess-Zumino term for superfivebrane, as we shall show later. However, these constructions raise a number of questions, among which is the interpretation of the new coordinates involved in the action.

(vi) Once the five-form generator $Z^{\mu_1 \cdots \mu_5}$ is included in the algebra, it is clear that one has to also include the two-form generator $Z^{\mu\nu}$, as can be seen from the $\{Q, \{Q, Q\}\}$ Jacobi identity and the γ-matrix identity (18). The reverse is not true, i.e. one can have the two-form generator without having to introduce the one- and/or five-form generators, in view of the γ-matrix identity (17). In fact, the super one- and/or five-form generators can be contracted away consistently.

Note in particular that the generators Z^A decouple from the algebra if we set $\lambda = \tau = 0$ and redefine the translation generator as $P_\mu + Z_\mu \equiv P'_\mu$. While it may be thought that Z^μ can always be redefined away, there are some global subtleties in doing so, and at least in the case of 10D superstrings, they have an interesting role to play in the description of the string winding states [22].

(vii) The fact that the super five-form generator requires the presence of the super two-form generator is related to the fact that a dual formulation of $D = 11$ supergravity containing only the six-form potential is not possible [19, 14]. The coexistence of the super two- and five-form generators in the M-algebra on the other hand, suggests a formulation of $D = 11$ supergravity theory in which both the three-form and the six-form potentials are used. However, a duality relation has to be imposed on the relevant field strengths, which then leads to non-localities [20].

(viii) Contracting away the super five-form generators yields the algebra of [9]. Contracting

away the super one-form generator as well, one obtains the algebra of [8]. Setting equal to zero $Z^{\alpha\beta}$ in addition gives the result of [6]. Keeping only the generators $Z^\alpha, Z^{\mu\nu}$ and $Z^{\mu_1\cdots\mu_5}$ gives the algebra studied in [13].

(ix) The fermionic generators Z^α and $Z^{\alpha_1\cdots\alpha_5}$ commute with all the other generators, except Lorentz generators.

(x) Dimensional reduction of the algebra (22) to ten and lower dimensions is expected to produce similar algebras for super p-branes existing in those dimensions. Aspects of these reductions will be treated elsewhere. It should be noted, however, that the Type IIB Poincaré superalgebra in 10D, as well as its M-algebra extension, if any, cannot be obtained in this way.

We now turn to the issue of Wess-Zumino terms based on the algebra (22). First of all we observe that the super four-form (15) is closed within the full algebra, and writing $H_4 = dC_3$, we find [8]

$$C_3 = -\tfrac{1}{6}e^\mu \wedge e^\nu \wedge e_{\mu\nu} - \tfrac{3}{20}e^\mu \wedge e^\alpha \wedge e_{\mu\alpha} + \tfrac{1}{30}e^\alpha \wedge e^\beta \wedge e_{\alpha\beta} \ . \tag{26}$$

As shown in [8], taking the standard $D = 11$ supermembrane action of [15], but using the pull-back of this C_3 as Wess-Zumino term, one finds an alternative formulation of the supermembrane, that generalizes a similar construction for the superstring due to Siegel [12]. In doing so, one introduces coordinates for all the generators, including ϕ^{AB}, but the dependence on the new coordinates comes as a total derivative term. This is due to the fact that the exterior derivative of C_3 defined in [15] and that of C_3 defined above give the same result, namely the super-four form (15), which in turn has no components along the new directions. Computing the Noether symmetry algebra corresponding to the full left group action, one directly finds the full algebra of these generators, without the occurrence of boundary terms in the Noether current that arise in a formulation with the usual non-manifestly supersymmetric version of the Wess-Zumino term.

To construct a Wess-Zumino term for the superfivebrane in eleven dimensions, we need a super seven-form. The obvious guess would be (16), but that form is not closed, as mentioned earlier. The γ-matrix identity (18) suggests the way to modify (16) to obtain a closed super seven-form as follows:

$$H_7 = \tfrac{1}{5!} \, e^{\mu_1} \wedge \cdots e^{\mu_5} \wedge e^\alpha \wedge e^\beta \, \gamma_{\mu_1\cdots\mu_5\alpha\beta} + H_4 \wedge C_3 \ , \tag{27}$$

Using the equations (21) and the identity (18), we see that indeed $dH_7 = 0$. Furthermore, writing $H_7 = dC_6$, we find that C_6 is given by

$$
\begin{aligned}
C_6 = \quad & \tfrac{1}{5!\times 77} \big(-\tfrac{77}{3} \, e^{\mu_1} \wedge \cdots \wedge e^{\mu_5} \wedge e_{\mu_1\cdots\mu_5} + \tfrac{281}{6} \, e^{\mu_1} \wedge \cdots \wedge e^{\mu_4} \wedge e^\alpha \wedge e_{\mu_1\cdots\mu_4\alpha} \\
& + \tfrac{104}{3} \, e^\mu \wedge e^\nu \wedge e^\rho \wedge e^\alpha \wedge e^\beta \wedge e_{\mu\nu\rho\alpha\beta} - \tfrac{47}{6} \, e^\mu \wedge e^\nu \wedge e^\alpha \wedge e^\beta \wedge e^\gamma \wedge e_{\mu\nu\alpha\beta\gamma} \\
& + 5 \, e^\mu \wedge e^{\alpha_1} \wedge \cdots \wedge e^{\alpha_4} \wedge e_{\mu\alpha_1\cdots\alpha_4} - \tfrac{5}{9} \, e^{\alpha_1} \wedge \cdots \wedge e^{\alpha_5} \wedge e_{\alpha_1\cdots\alpha_5}
\end{aligned}
$$

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$$-\tfrac{131}{3}\, e^\mu \wedge e^\nu \wedge e^\rho \wedge e^\alpha \wedge e_{\mu\nu} \wedge e_{\rho\alpha} + \tfrac{50}{3}\, e^\mu \wedge e^\nu \wedge e^\alpha \wedge e^\beta \wedge e_{\mu\nu} \wedge e_{\alpha\beta}$$

$$+\tfrac{20}{3}\, e^\mu \wedge e^\alpha \wedge e^\beta \wedge e^\gamma \wedge e_{\mu\alpha} \wedge e_{\beta\gamma} \;)\;. \tag{28}$$

Interestingly, the super-form (27) was considered long ago [14] within the framework of a sub-algebra of (22) in which only Z^α, $Z^{\mu\nu}$ and $Z^{\mu_1\cdots\mu_5}$ are kept in addition to the super Poincaré generators.

The same super-form was also considered in [20] within the framework of the usual 11D Poincaré superalgebra. It was shown in [20] that defining in curved superspace

$$H_4 \;=\; \tfrac{1}{4} e^a \wedge e^b \wedge e^\alpha \wedge e^\beta \; \gamma_{ab\alpha\beta} + e^{a_1} \wedge \cdots \wedge e^{a_4} \; H_{a_1\cdots a_4}\;, \tag{29}$$

$$H_7^{(0)} \;=\; \tfrac{1}{5!} e^{a_1} \wedge \cdots \wedge e^{a_5} \wedge e^\alpha \wedge e^\beta \; \gamma_{a_1\cdots a_5\alpha\beta} + e^{a_1} \wedge \cdots \wedge e^{a_7} \; \epsilon_{a_1\cdots a_7 c_1\cdots c_4} \; H^{c_1\cdots c_4}\;, \tag{30}$$

one finds, via the Bianchi identities $dH_4 = 0$ and $dH_7^{(0)} = H_4 \wedge H_4$, the correct equation of motion for $D = 11$ supergravity. However, as was emphasized in [20], if one wishes to work with a super-six form potential C_6 alone, then one has a non-local relationship between $H_7^{(0)}$ and C_6.

Turning to the super six-form (28) which is defined in the M-extended Poincaré superspace, we can write down a Wess-Zumino term for a superfivebrane as follows

$$I_{WZ} = \int \underline{C}_6\;, \tag{31}$$

where $\underline{C}_6$ is the pull-back of C_6. We use a notation in which the underlining of a target space form indicates its pull-back. This action is manifestly invariant under the M-group transformations, including supersymmetry. However, $H_7 = dC_6$ equals an expression that contains C_3 as shown in (27), which in turn has nonvanishing components in the e^{AB} directions, as shown in (26). Therefore, the Wess-Zumino action (31) contains the coordinates ϕ_{AB} associated with the generators Z^{AB} such that they are not confined to a total derivative term. This is in contrast to the supermembrane case where all the dependence on the new coordinates is contained in a total derivative term [8].

Using the super three-form C_3 of the standard $D = 11$ supermembrane action [15] in the definition of H_7, on the other hand, would yield a closed super seven-form that strictly lives in the usual Poincaré superspace. However, the resulting Wess-Zumino term would not be manifestly supersymmetric.

In the context of usual Poincaré superspace, let us focus our attention to the case of purely bosonic target space background, and consider the two-form gauge transformations $\delta C_3 = d\lambda_2$. From $H_7 = H_7^{(0)} + H_4 \wedge C_3$, noting that $H_7^{(0)}$ is invariant, one sees that C_6 must transform as $\delta C_6 = C_3 \wedge d\lambda_2$ [20]. Of course, I_{WZ} given in (31) is not invariant under these transformations. However, it has been observed that [21], since the worldvolume fields of the superfivebrane include

a fundamental two-form B_2, there is a way to write down a manifestly tensor-gauge invariant Wess-Zumino term, namely $I_{WZ} = \int(\underline{C}_6 + dB_2 \wedge \underline{C}_3)$, with B_2 transforming as $\delta B_2 = \lambda_2$.

Turning to the case of superfivebrane in the context of the M-extended Poincaré superspace, we can consider, in analogy with the case discussed above, the following Wess-Zumino term:

$$I'_{WZ} = \int (\underline{C}_6 + dB_2 \wedge \underline{C}_3) \ , \tag{32}$$

where, we recall that C_6 and C_3 are given in (28) and (26). This term, just as in the case of (31) discussed earlier, contains coordinates other than X^μ and θ^α, which are not contained in a total derivative term. The target space is flat or curved M-extended superspace. In the latter case relations similar (29) and (30) can be utilized. However, since super-form coordinates occur in the action, it is not altogether clear what this implies for $D = 11$ supergravity, and whether it can lead its dual formulation in a novel way.

Going back to the issue of Wess-Zumino terms, it should be emphasized that one can construct a number of distinct and manifestly supersymmetric Wess-Zumino terms, within the context of the M-algebra, by taking various Lorentz-invariant combinations of the left-invariant super-one forms $L^{\hat{A}}$. However, presumably not all of these terms are relevant for the seeked superfivebrane action with the right properties.

To illustrate the point about the variety of ways in which a superfivebrane Wess-Zumino term can be constructed in our framework, we present an example which is particularly interesting because it makes use of the stringy coordinates Z^A. Consider the super seven-form

$$H''_7 = H_4 \wedge H_3 \ , \tag{33}$$

where H_4 is defined in (15) and H_3 in (24). Since H_4 and H_3 are closed, so is H_7. Moreover, writing $H'_7 = dC'_6$, we have

$$C''_6 = C_3 \wedge H_3 \ , \tag{34}$$

(up to an irrelevant closed form) where C_3 is given in (26) and H_3 in (24). Unlike in the case of H_7 which can be formulated in ordinary Poincaré superspace, the existence of H''_7 requires the M-extended superspace based on the M-algebra, or a suitable subalgebra thereof. Using (34), we can construct a third kind of Wess-Zumino term given by

$$I''_{WZ} = \int \underline{C}''_6 \ . \tag{35}$$

This is manifestly supersymmetric and tensor-gauge invariant. However, whether it can be used in the construction of a sensible superfivebrane action, and if so, exactly which $D = 11$ supersymmetric field theory it may describe, remains to be seen.

So far, we have discussed the superfivebrane Wess-Zumino terms. As far as the kinetic term is concerned, matters are somewhat more complicated. Even in the case of a minimal target superspace without any new coordinates, the full kinetic term is not known. For the progress made in this front, and a discussion of various related matters, see [21, 22, 23, 24, 25]. The problem is further complicated in the M-extended superspace, because of the presence of new coordinates. To avoid the propagation of any unwanted degrees of freedom, one has to find new kinds of fermionic and bosonic local symmetries, analogous to the more familiar κ-symmetry.

Acknowledgements

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